Pre-class Warm-up!!!

Consider a system $x' = Ax$ where A has an eigenvector $\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$ with eigenvalue $\lambda = 1 + 3i$.

We get solutions $\begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{(1+3i)t}$ and its complex conjugate.

$$\begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{t}(\cos 3t + i \sin 3t)$$

$$= e^{t}\begin{bmatrix} \cos 3t + i \sin 3t \\ 2\cos 3t + \sin 3t + i(2\sin 3t - \cos 3t) \end{bmatrix}$$

3. Which of the following are solutions?

✓a. $e^{t}\begin{bmatrix} \cos 3t + i \sin 3t \\ 2\cos 3t + \sin 3t + i(2\sin 3t - \cos 3t) \end{bmatrix}$

b. $e^{t}\begin{bmatrix} \cos 3t + i \sin 3t \\ 2\cos 3t + i \sin 3t \end{bmatrix}$

4. Which of the following are solutions?

a. $e^{t}\begin{bmatrix} \cos 3t \\ 2\sin 3t \end{bmatrix}$

✓b. $e^{t}\begin{bmatrix} \cos 3t \\ 2\cos 3t + \sin 3t \end{bmatrix}$

8.1 Matrix exponentials and linear systems

We learn

- a new approach to solving homogeneous systems
- how to solve homogeneous systems when the matrix is not diagonalizable

Vocabulary:

- fundamental matrix of a linear system
- Matrix exponential
- Nilpotent matrix = matrix A with $A^n = 0$ for some n.

A different approach when the matrix is not diagonalizable is described in section 7.6 in terms of generalized eigenvectors and generalized eigenspaces. This is done in more advanced courses like Math 4242.

e.g. $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Page 479 question 2

Find a fundamental matrix of the system and solve the initial value problem, where

$$\underline{x}' = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \underline{x}, \quad \underline{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution: Char. poly $= (2-\lambda)^2 - 4 = \lambda(\lambda-4)$

Find e-vectors: $\lambda = 0$. Null $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ has basis $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 4$, Null $\begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix}$ has basis $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

General solution $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}}_{\text{This is a fundamental matrix } \Phi} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underline{x}(t)$$

Impose the initial condition

$$\underline{x}(0) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{4} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 \\ 5/4 \end{bmatrix}$$

Symbolically $x(t) = \Phi(t) \underline{c}$

$x(0) = \Phi(0) \underline{c}$, $\underline{c} = \Phi(0)^{-1} x(0)$

Finally $x(t) = \Phi(t) \Phi(0)^{-1} \underline{x}(0)$

A fundamental matrix for the system is a matrix whose columns are a basis for the solution space.

It is not unique.

Definition.
A fundamental matrix for a system $x' = Ax$ is a matrix Φ whose columns form a basis for the space of solutions to the system.

Theorem. Let Φ be a fundamental matrix for $x' = Ax$, and suppose there is an initial condition $x(0) = x_0$. Then the solution to this initial value problem is

$$x(t) = \Phi(t)\Phi^{-1}(0)\,x_0$$

Definition.
The exponential of an n x n matrix A is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

This always converges. $\frac{A^d}{d!}$ is a matrix where in each entry we have a sum of products of d of the entries of A. Dividing by $d!$ we get convergence in each entry of e^A.

Properties:

1. e^{A+B} $= e^A e^B$ provided A and B commute. We need $(A+B)^d = \sum_{r=0}^{d} \binom{d}{r} A^r B^{d-r}$ to work and this works if $AB = BA$.

2. e^0 $= I$

3. (d/dt) e^{At} $= A e^{At}$

by doing $\frac{d}{dt}\left(I + At + \frac{1}{2!}A^2t^2 + \cdots\right) = 0 + A + A^2t + \cdots = A e^{At}$

Page 480 question 23. Show that the matrix A is nilpotent. Find e^{At} where

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$A^2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \underbrace{\frac{1}{3!}(At)^3}_{=0}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} t & -t & -t \\ t & -t & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & -2t^2 \\ 0 & 0 & -2t^2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 2. The solution to $x' = Ax$, $x(0) = x_0$ is $\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0$

Proof. $\frac{d}{dt}\left(e^{At}\underset{\sim}{x}_0\right) = Ae^{At}x_0$ so

$x = e^{At}x_0$ satisfies the equation and $x(0) = x_0$.

Like page 480 question 25:
Solve $x' = Ax$, $x(0) = (1,2,3)$ with A as in question 23.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Completed after class:

$$x(t) = e^{At}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

computed on the last page

$$= \begin{bmatrix} 1-4t-3t^2 \\ 2+2t-3t^2 \\ 3 \end{bmatrix}$$

Theorem 3. $e^{At} = \Phi(t) \cdot \Phi(0)^{-1}$

Proof. Both sides appear in solutions to $x' = Ax$, $x_0 = x(0)$ for every x_0!

$x = e^{At} x_0 = \Phi(t)\Phi(0)^{-1} x_0$ $\square$

Like question 9.
Compute e^{At} when $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$

Solution: The fundamental matrix for $x' = Ax$ is $\Phi = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}$ and

$\Phi(0)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

and $e^{At} = \Phi\,\Phi(0)^{-1} = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

$= \begin{bmatrix} \frac{1+e^{4t}}{2} & \frac{1-e^{4t}}{4} \\ 1-e^{4t} & \frac{1+e^{4t}}{2} \end{bmatrix}$

Question 26.
Solve the IVP $x' = \begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix} x$, $x(0) = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$

Note $\begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix}$ is not diagonalizable and the e-val / e-vec approach doesn't help.
Also $\begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix}$ is not nilpotent.

Solution
We calculate $e^{\begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix} t}$

$= e^{\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} t + \begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t} = e^{\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} t} \cdot e^{\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t}$

because $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix}$ commute.

We get $\begin{bmatrix} e^{7t} & 0 \\ 0 & e^{7t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 11t & 1 \end{bmatrix}$

$= \begin{bmatrix} e^{7t} & 0 \\ 11te^{7t} & e^{7t} \end{bmatrix}$

$e^{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}} = I + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 + \frac{1}{3!}\begin{bmatrix} & \\ & \end{bmatrix}^3 + \cdots$

$= \begin{bmatrix} 1 + a + \frac{1}{2!}a^2 + \cdots & 0 \\ 0 & 1 + b + \frac{1}{2!}b^2 + \cdots \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$

Pre-class Warm-up!!!

What is e^{At} when $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$?

a. $\begin{bmatrix} 3 & t \\ 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix}$ ✓

d. $\begin{bmatrix} e^{3t} \end{bmatrix}$

e. None of the above.

Another question: what is e^{Bt} when

$B = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$